LIOUVILLE THEOREM WITH PARAMETERS: ASYMPTOTICS OF CERTAIN RATIONAL INTEGRALS IN DIFFERENTIAL FIELDS

MAŁGORZATA STAWISKA

Abstract. We study asymptotics of integrals of certain rational functions that depend on parameters in a field K of characteristic zero. We use formal power series to represent the integral and prove certain identities about its coefficients following from generalized Vandermonde determinant expansion. Our result can be viewed as a parametric version of a classical theorem of Liouville. We also give applications.

1. Integrating rational functions over differential fields

Let $\mathcal{D}$ be a differential field. This means that $\mathcal{D}$ is a field with an additional mapping $' : \mathcal{D} \mapsto \mathcal{D}$ (differentiation) satisfying the following two conditions: $(u+v)' = u' + v'$ and $(uv)' = u'v + uv'$ for all $u, v \in \mathcal{D}$. The set $K = \{u \in \mathcal{D} : u' = 0\}$ is a subfield of $\mathcal{D}$, called the field of constants. We use the following terminology, adapted from [Ris]: Let U be a universal (differential) extension of $\mathcal{D}$. For $u \in U$, u and $\mathcal{D}(u)$ are said to be simple elementary over $\mathcal{D}$ iff one of the following conditions holds: (1) u is algebraic over $\mathcal{D}$; (2) There is a v in $\mathcal{D}$, $v \neq 0$ such that $v' = vu'$ (we will write equivalently $u = \log v$); (3) There is a v in $\mathcal{D}$, $v \neq 0$ such that $u' = uv'$. (or equivalently $u = \exp v$).

We say that $\mathcal{F}$ and any $w \in \mathcal{F}$ is elementary over $\mathcal{D}$ if $\mathcal{F} = \mathcal{D}(u_1, ..., u_n)$ for some n, where each u_i is simple elementary over $\mathcal{D}(u_1, ..., u_{i-1})$, $\quad i = 1, ..., n$.

The following theorem dates back to J. Liouville (cf. [Ris]):

2010 *Mathematics Subject Classification.* 13F25,41A58,12Y99.
Key words and phrases. Liouville theorem, differential fields, integrals of rational functions.

Theorem 1. *Let $\mathcal{D}$ be a differential field, $\mathcal{F}$ elementary over $\mathcal{D}$. Suppose $\mathcal{D}$ and $\mathcal{F}$ have the same constant field K. Let $g \in \mathcal{F}$, $f \in \mathcal{D}$ with $g' = f$. Then $g = v_0 + \sum c_i \log v_i$, where v_0, v_i are elements of $\mathcal{D}$ and c_i are elements of K.*

We will write $f = g$ as equivalent to $g' = f$.

Example and notation: Let K be an arbitrary field of characteristic zero and z be transcendental over K. We introduce differentiation in the polynomial ring $K[z]$ by taking $z' = 1$ and $a' = 0$ for all $a \in K$ (the standard differentiation of polynomials in one variable). The field $K(z)$ of rational fractions of $K[z]$ is a differential field when we extend $(z^{-1})' = -z^{-1}z^{-1}$, and K is its field of constants.

With K as in the example, we will consider the ring $K[[1/z]$ of the following formal series: $\sum_{n=0}^{\infty} a_n z^{-n}$ with $a_n \in K$. The differentiation ' can be extended term-by-term as a map of $K[[1/z]]$ to itself. One can also define a valuation $o : K[[1/z]] \mapsto \mathbb{N} \cup \{\infty\}$ as follows (cf. [VS], discussion before Proposition 2.3.16): $o(f) = \min\{n : a_n \neq 0\}$ and $o(0) = \infty$.
Consider a a square-free polynomial $Q(z) = z(z - a_1)...(z - a_q)$ with $a_n \in K$. Our result can be now formulated as follows:

Theorem 2. *(a) In an elementary field $\mathcal{F}$ over $K(z)$, consider the elements $g = \int f \in \mathcal{F}$ with $f = 1/Q$, where $Q(z) = z(z - a_1)...(z - a_q)$ is a square-free polynomial with $a_1, ..., a_q \in K$, $\quad q \geq 1$. The set G of all such elements is in a bijective correspondence with a subset of $K[[1/z]]$.*
(b) For (the image of) a $g = \int 1/Q$ we have $o(g) = q$, where $q = \deg Q + 1$.

In the proof of this theorem we will apply the following identities:

Lemma 1.

$$\frac{1}{Q'(0)} + \frac{1}{Q'(a_1)} + ... + \frac{1}{Q'(a_q)} = 0,$$

$$\frac{a_1}{Q'(a_1)} + ... + \frac{a_q}{Q'(a_q)} = 0,$$

...

$$\frac{a_1^{q-1}}{Q'(a_1)} + ... + \frac{a_q^{q-1}}{Q'(a_q)} = 0,$$

$$\frac{a_1^q}{Q'(a_1)} + ... + \frac{a_q^q}{Q'(a_q)} = 1,$$

$$\frac{a_1^{q+l}}{Q'(a_1)} + ... + \frac{a_q^{q+l}}{Q'(a_q)} = S_l(a_1, ..., a_q),$$

where S_l is the complete homogeneous polynomial of degree l, symmetric in its variables, i.e., $S_l(X_1, ..., X_n) = \sum_{1 \le i_1 \le ... \le i_l \le n} X_{i_1} ... X_{i_l}$ for $l = 1, 2, ...$

Proof. (of Lemma) We use properties of the Vandermonde determinant:

$$V_n(x_1, ..., x_n) = \begin{vmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & x_n & \dots & x_n^{n-1} \end{vmatrix}$$

Recall that $V_n(x_1, ..., x_n) = \prod_{1 \le i < j \le n}(x_j - x_i)$ and $V_{n+1}(x_1, ..., x_{n+1}) = (-1)^n \prod_{i=1}^{n}(x_i - x_{n+1}) V_n(x_1, ..., x_n)$. More generally, one can consider

$$V_{n,l}(x_1, ..., x_n) = \begin{vmatrix} 1 & x_1 & \dots & x_1^{n-1+l} \\ 1 & x_2 & \dots & x_2^{n-1+l} \\ \dots & \dots & \dots & \dots \\ 1 & x_n & \dots & x_n^{n-1+l}, \end{vmatrix}$$

where $l = 1, 2, ...$ Then $V_{n,l}(x_1, ..., x_n) = V_n(x_1, ..., x_n) \cdot S_l(x_1, ..., x_n)$, where S_l is the complete homogeneous polynomial of degree l in $x_1, ..., x_n$ ([Ma], formula I.3.1).

Note that for $Q(z) = z(z - a_1)...(z - a_q)$ one has $Q'(0) = (-1)^q \prod_{i=1}^q a_1...a_q$, $Q'(a_i) = \prod_{j \ne i}(a_i - a_j)$. To prove the first stated identity, let us make $V_{q+1}(0, a_1, ..., a_q)$ the common denominator of the left hand side. Then $1/Q'(0) = (-1)^q V_q(a_1, ..., a_q)/V_{q+1}(0, a_1, ..., a_q)$ and $1/Q'(a_i) = (-1)^{n-i-1} \prod_{k \ne i} a_k \prod_{k,j \ne 1, k<j}(a_k - a_j)$, so the numerator is the cofactor of the element 1 in the i-th row of V_q. Thus in the summation of all terms corresponding to different roots of Q the numerator is V_q minus its Laplace expansion along the column of 1's, which equals 0. The same argument proves the second identity: in the common denominator we now have $V_q(a_1, ..., a_q)$ and the numerator is a Vandermonde determinant of size $(q-1) \times (q-1)$ minus its Laplace expansion along the column of 1's . In the sum of $\frac{a_i^k}{Q'(a_i)}, \quad k \le q-1$, the numerator is the Laplace expansion of

$$\begin{vmatrix} 1 & a_1 & ...a_1^{q-1} & 1 \\ \dots & \dots & \dots & \dots \\ 1 & a_n & ...a_q^{q-1} & 1 \end{vmatrix}$$

along the column containing terms of the type a^k, and in the sum of $\frac{a_i^{q+l}}{Q'(a_i)}$ the numerator is the Laplace expansion of $V_{q,l}(a_1,...a_q)$ with respect to the last column, which is a product of $V_q(a_1,...a_q)$ by the complete homogeneous symmetric polynomial $S_l(a_1,...a_q)$ of degree $l > 0$.

□

Remark 1. The last identity was obtained in a different way as Theorem 3.2 in [Co], where it is also traced back to C.G.J. Jacobi.

Proof. (of Theorem 2)

First note that for $a \in K$ we can identify $(z-a)^{-1}$ with $\sum_{n=1}^{\infty} a^n z^{-(n+1)}$. More generally, if $Q(z) = z(z-a_1)...(z-a_q)$ is a square-free polynomial with $a_n \in K$ and $P \in K[z]$, then partial fraction decomposition gives $P/Q = c_0/z + c_1/(z-a_1) + ... + c_q/(z-a_q)$ with $c_n = P(a_n)/Q'(a_n), \quad n = 1,...,q$ (cf. [Tr]) and P/Q can also be identified with an element of $K[[1/z]]$. It follows that $\log(1-a/z)$ can be identified with $\sum_{n=1}^{\infty}((-1)^n a^n/n)z^{-n}$. Let now $g = \int(1/Q)$ with Q as above. Then $g = b_0 + \frac{1}{Q'(0)}\log z + \frac{1}{Q'(a_1)}\log(z-a_1) + ... + \frac{1}{Q'(a_q)}\log(z-a_q)$ in $\mathcal{F}$ with b_0 in K. Identifying each $\log(z-a_j), \quad j = 1,...,q$ with an appropriate formal series in $K[[1/z]]$ as above and adding the results, we get $g = \sum_{n=0}^{\infty} b_n z^{-n}$. By the lemma, $b_1 = ... = b_{q-1} = 0$, $b_q = 1/q$ and $b_{q+l} = S_l(a_1,...,a_q)/(q+1)$, where S_l is the complete homogeneous symmetric polynomial of degree l, $l = 1,2,....$ To ensure this is the only possible series in $K[[1/z]]$ that can be identified with $g = \int(1/Q)$, note that any such series should be symmetric with respect to $a_1,...,a_q$. Theorem 9.3 in [LS] says the following: If a formal series F in the variables $X_1,...,X_q,Y$ is symmetric with respect to $(X_1,...,X_q)$, then $F = \Phi(\sigma_1,...,\sigma_q,Y)$, where Φ is a formal series in the variables $X_1,...,X_q,Y$ and $\sigma_1,...,\sigma_q$ are elementary symmetric polynomials in $X_1,...,X_q$. Moreover, the series Φ is unique. Uniqueness of our $\sum_{n=0}^{\infty} b_n z^{-n}$ follows, because $S_l(X_1,...,X_q)$ can be expressed as polynomials in $\sigma_1,...,\sigma_q$. Hence also $o(g) = q$. □

2. Applications

A particular case of our Theorem 2 is Proposition 1 in [GS], which was proved for $K = \mathbb{C}$ and $a_j = \zeta^j a_1, j = 1,...,q$, where ζ is a primitive root of unity of order q. The convergence of $\int 1/(z(z-a_1)...(z-a_q)) \to -1/(qz^q)$ as $a_1,...,a_q \to 0$ (which is in fact uniform for $|z| > R$) is important in constructing approximate Fatou coordinates for analytic maps f in a neighborhood of an $f_0(z) = z + z^{q+1} + ...$ with $q > 1$. These are coordinates in which f looks like a translation. The first

step in constructing Fatou coordinate for f_0 consists in lifting f_0 to a neighborhood of infinity by the coordinate change $z \mapsto -1/(qz^q)$. We considered f belonging to an one-parameter family of polynomials $P_\lambda(z) = \lambda z + z^2$ with $\lambda_0 = e^{2\pi i p/q}$ and $\lambda = e^{2\pi i(p/q+u)}$, with p, q coprime integers and u in a sufficiently small neighborhood of 0 in $\mathbb{C}$. We started the construction of near-Fatou coordinate by applying the transformation $w(z) = \int_{z_0}^{z}(1/Q(u,\zeta))d\zeta$, where $Q(u,z)$ is the Weierstrass polynomial for $P_\lambda^{\circ q}(z) - z$. As u is small, the non-zero solutions of $Q(u,z) = 0$ are also small. Because of convergence of integrals, the coordinates obtained for P_λ depend continuously on u. For more details and references see [GS].

Another application is a generalization of the well-known formula for electrostatic potential of a dipole located at $z = 0$ (cf. [Ne]): consider two charges $1/a$ and $-1/a$ placed respectively at $z = 0$ and $z = a$. Then their combined electrostatic potential is $(1/a)\log z - (1/a)\log(z-a)$, which tends to $-1/z$ as $a \to 0$ (this follows easily from the definition of derivative of the logarithmic function). In the setting of this paper, there are charges $1/Q'(0), 1/Q'(a_1), ..., 1/Q'(a_q)$ placed at $0, a_1, ..., a_q$ (with $Q(z) = z(z-a_1)...(z-a_q)$), and it follows from Theorem 2 that the potential tends to $-1/(qz^q)$ as $a_1, ..., a_q \to 0$.

Acknowledgment: The main idea of the paper occured while I was working with Estela Gavosto on [GS]. I thank her for many useful conversations.

References

[Co] E. F. Cornelius, Jr.: Identities for complete homogeneous symmetric polynomials, preprint, 2009, available at: http://www.scribd.com/doc/16010484/Identities-for-Complete-Homogeneous-Symmetric-Polynomials

[GS] E. Gavosto, M. Stawiska: Parabolic explosions in families of complex polynomials, preprint, 2010, available at: http://www.math.ku.edu/˜mfriedland/

[LS] S. Łojasiewicz, J. Stasica: *Analiza formalna i funkcje analityczne*, Wydawnictwo Uniwersytetu Jagiellońskiego [*Formal Analysis and Analytic Functions*, Jagiellonian University Press], Kraków, 2005

[Ma] Macdonald, I. G.: *Symmetric functions and Hall polynomials. Second edition. With contributions by A. Zelevinsky.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[Ne] Needham, Tristan: *Visual complex analysis.* The Clarendon Press, Oxford University Press, New York, 1997

[Ris] Risch, Robert H.: The problem of integration in finite terms. *Trans. Amer. Math. Soc.* 139 (1969), 167–189

[VS] Villa Salvador, Gabriel Daniel: *Topics in the theory of algebraic function fields.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 2006.

[Tr] B. Trager: Algebraic factoring and rational function integration, *Proceedings of the 1976 ACM Symposium on Symbolic and Algebraic Computation*, manuscript

Department of Mathematics, University of Kansas, Lawrence, KS 66045

E-mail address: stawiska@ku.edu